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Explicit Darboux transformations of arbitrary order for generalized time-dependent Schrödinger equations

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Abstract
We construct Darboux transformations of arbitrary order for a generalized, linear, time-dependent Schrödinger equation, special cases of which correspond to time-dependent Hamiltonians coupled to a magnetic field, with position-dependent mass and with weighted energy. Our Darboux transformation reduces correctly to these known cases and also to new, generalized Schrödinger equations. Furthermore, fundamental properties of the conventional Darboux transformation are maintained, such as factorization of the nth order transformation into first-order transformations and existence of a reality condition for the transformed potentials.

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1. Introduction

The Darboux transformation is one of the major tools for generating solvable cases of linear Schrödinger equations. Its key feature is the conversion of differential equations into differential equations of the same form. Being the application of a particular linear differential operator to the solution of a differential equation, the Darboux transformation does not involve coordinate changes, which makes it essentially different from other popular methods for generating solutions, such as, e.g., Lie symmetry transformations. In its first version [4], the Darboux transformation was applicable to equations of stationary Schrödinger form including a first derivative term. At some point it was then discovered that the applicability of the Darboux transformation could be extended to the fully time-dependent case [5], and that it was equivalent to the formalism of supersymmetric quantum mechanics [3]. After that, several generalizations of the stationary and the time-dependent Schrödinger equation were found that admitted particular Darboux transformations, all with a similar form and similar properties, such as, e.g., their explicit form or their equivalence to factorization formalisms within the supersymmetry context. Examples of such equations that allow for Darboux transformations include the Schrödinger equation with first-order derivatives [8], with position-dependent mass [6] and with weighted energy [9, 10]. Since the Darboux transformations of these particular equations present several similarities, it seems likely that they are special cases of a more general Darboux transformation. The construction of this generalized Darboux transformation is the purpose of the present paper. More precisely, we will consider a time-dependent Schrödinger equation with a first-order spatial derivative term and nonconstant, independent coefficients. This equation, comprising the cases mentioned above, will be shown to have a Darboux transformation of arbitrary order. Furthermore, this transformation maintains the main properties of its conventional counterpart (i.e., the Darboux transformation for the standard Schrödinger equation), such as factorizability into first-order Darboux transformations and existence of a reality condition for the transformed potential [2]. In summary, the present paper gives a characterization of Darboux transformations for linear, time-dependent Schrödinger equations. In section 2, we summarize facts about the Darboux transformation for the Schrödinger equation and give some examples of generalizations with the corresponding Darboux transformations. In section 3, we state our results, and in section 4 we show them to reduce correctly to the well-known, conventional case. Section 5 is devoted to the proof of our results, and in section 6 we illustrate our considerations by a simple example.

2. Preliminaries

For the sake of completeness let us state basic facts about the Darboux transformation for the Schrödinger equation and a few of its generalizations.

The Darboux transformation. Consider the time-dependent Schrödinger equation

$$\mathrm{i}\Psi_t + \frac{1}{2m}\Psi_{xx} - V\Psi = 0, \tag{1}$$

where m stands for the constant mass, and $V = V(x,t)$ is the potential. The nth order Darboux transformation of a solution Ψ to (1) is defined as

$$D_{n,(u_j)}(\Psi) = L\frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}, \tag{2}$$

where $L = L(t)$ is an arbitrary, purely time-dependent function, the family (u_j) of n auxiliary solutions to (1) are such that $(u_1, u_2, \ldots, u_n, \Psi)$ is linearly independent, and $W_{n,(u_j)}$, $W_{n,(u_j),\Psi}$ denote the Wronskians of (u_j) and of (u_j, Ψ), respectively. Note that these Wronskians depend on both variables x and t, but for the sake of brevity we have left out these variables as arguments in (2). The function $\hat{\Psi} = D_{n,(u_j)}(\Psi)$ solves the time-dependent Schrödinger equation

$$\mathrm{i}\hat{\Psi}_t + \frac{1}{2m}\hat{\Psi}_{xx} - U\hat{\Psi} = 0, \tag{3}$$

where the potential U reads

$$U = V + \mathrm{i}\frac{L'}{L} - \frac{1}{m}\left[\log\left(W_{n,(u_j)}\right)\right]_{xx}. \tag{4}$$

Thus, the nth order Darboux transformation establishes a relation between the TDSEs (1) and (3) and has the following fundamental properties:

- The nth order Darboux transformation factorizes, i.e. it can be written as an iteration of n first-order Darboux transformations [1].
- There is a condition on the function L, such that the transformed potential U becomes a real-valued function [2].

In the present paper, we shall show that these properties persist in generalizations of the Darboux transformations for linear Schrödinger equations.

Generalizations. We now mention three typical generalizations of the time-dependent Schrödinger equation that admit Darboux transformations.

- The Schrödinger equation with first-order derivatives [8]:

$$\mathrm{i}\Psi_t + \Psi_{xx} + 2\mathrm{i}R\Psi_x + (\mathrm{i}R_x - V)\,\Psi = 0, \tag{5}$$

  where $R = R(x,t)$ is arbitrary, and $V = V(x,t)$ denotes the potential. The Hamiltonian associated with equation (5) has the form of a three-dimensional Hamiltonian coupled to a magnetic field. The nth order Darboux transformation, $\mathcal{D}_{n,(u_j)}(\Psi)$, for this equation has the form

$$\mathcal{D}_{n,(u_j)}(\Psi) = L\frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}},$$

  where $L = L(t)$ is arbitrary, $W_{n,(u_j)}$ and $W_{n,(u_j),\Psi}$ are the Wronskians of a family (u_j) of n auxiliary solutions to (5) and of the solution Ψ, respectively.
- The position-dependent mass Schrödinger equation [6]:

$$\mathrm{i}\Psi_t + \frac{1}{2m}\Psi_{xx} - \frac{m_x}{2m^2}\Psi_x - V\Psi = 0, \tag{6}$$

  where $m = m(x,t)$ stands for the nonconstant mass, and $V = V(x,t)$ is the potential. This equation allows for the following nth order Darboux transformation:

$$\mathcal{D}_{n,(u_j)}(\Psi) = L\left(\frac{1}{m}\right)^{\frac{n}{2}}\frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}.$$

  Here an analogous notation as in the previous point was employed.
- The stationary Schrödinger equation with weighted energy [9, 10]:

$$\Psi'' + (Eh - V)\Psi = 0, \tag{7}$$

  where $h = h(x)$ is arbitrary. Note that (7) can also be seen as a Schrödinger equation with linearly energy-dependent potential. Here we have the following nth order Darboux transformation:

$$\mathcal{D}_{n,(u_j)}(\Psi) = L\left(\frac{1}{h}\right)^{\frac{n}{2}}\frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}.$$

  Here L is a constant, and there is no dependence on t, since we are dealing with a stationary equation.

3. Summary of results

Consider the following generalized Schrödinger equation in (1+1) dimensions:

$$\mathrm{i}h\Psi_t + f\Psi_{xx} + g\Psi_x - V\Psi = 0, \tag{8}$$

where the indices denote partial differentiation, and all involved functions f, g, h and the potential V depend on the variables x and t. Note that h has nothing to do with Planck's constant $\hbar$. Note further that one of the coefficients is obsolete, as by division it can be absorbed into the remaining coefficients. However, throughout the following considerations we stick to the general form (8) of the Schrödinger equation, as it allows more easily for the derivation of special cases.

3.1. Darboux transformation

Let Ψ be a solution of equation (8), and let (u_j) be a family of n auxiliary solutions of equation (8), such that the family (u_j, Ψ) is linearly independent. Define the nth order Darboux transformation of Ψ as

$$\mathcal{D}_{n,(u_j)}(\Psi) = L\left(\frac{f}{h}\right)^{\frac{n}{2}} \frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}, \tag{9}$$

for an arbitrary function $L = L(t)$. The function, $\hat{\Psi} = \mathcal{D}_{n,(u_j)}(\Psi)$, is a solution of the generalized Schrödinger equation,

$$\mathrm{i}h\hat{\Psi}_t + f\hat{\Psi}_{xx} + g\hat{\Psi}_x - U\hat{\Psi} = 0. \tag{10}$$

The function, $U = U(x,t)$, is given explicitly by the following expression:

$$\begin{aligned} U = V + \mathrm{i}v'h\frac{L'}{L} - 2\sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}} W_{n,(u_j)}\right)\right]_x\right]_x \\ +2nf\left(F_{xx} + \frac{F_x}{2}\left[\log\left(\frac{f}{h}\right)\right]_t\right), \end{aligned} \tag{11}$$

where $\kappa = n(n-1)/2$, v' is the derivative of an arbitrary function $v = v(t)$ and $F = F(x,t)$ reads

$$F = -\int\left(\frac{g}{2f} + \frac{h_x}{4h} - \frac{f_x}{4f} + \frac{\mathrm{i}}{2}\sqrt{\frac{h}{v'f}}\left[\sqrt{v'}\int\sqrt{\frac{h}{f}}\,\mathrm{d}x\right]_t\right)\mathrm{d}x. \tag{12}$$

In the particular case of a first-order Darboux transformation, that is, for $n = 1$ in (9), the transformation and the transformed potential (11) simplify as follows:

$$\mathcal{D}_{1,u_1}(\Psi) = L\sqrt{\frac{f}{h}}\left(-\frac{(u_1)_x}{u_1}\Psi + \Psi_x\right) \tag{13}$$

$$U = V + \mathrm{i}v'h\frac{L'}{L} - 2\sqrt{fh}\left[\sqrt{\frac{f}{h}}[\log(u_1)]_x\right]_x + 2f\left(F_{xx} + \frac{F_x}{2}\left[\log\left(\frac{f}{h}\right)\right]_t\right). \tag{14}$$

In summary, the Darboux transformation (9) interrelates the solutions Ψ and $\mathcal{D}_{n,(u_j)}(\Psi)$ of the generalized Schrödinger equations (8) and (10), respectively.

3.2. *Chains of Darboux transformations and factorization*

The nth order Darboux transformation (9) can always be written as a chain (iteration) of n first-order Darboux transformations:

$$\mathcal{D}_{n,(u_j)}(\Psi) = \mathcal{D}_{1,v_n} \circ \mathcal{D}_{1,v_{n-1}} \circ \cdots \circ \mathcal{D}_{1,v_1}(\Psi), \tag{15}$$

where v_j, $j = 1, \ldots, n$, is an auxiliary solution of the $(j-1)$ th transformed generalized Schrödinger equation. More explicitly, relation (15) reads

$$\begin{aligned}\mathcal{D}_{n,(u_j)}(\Psi) = L &\left[\sqrt{\frac{f}{h}}\left(-\frac{(v_n)_x}{v_n} + \frac{\partial}{\partial x}\right)\right]\left[\sqrt{\frac{f}{h}}\left(-\frac{(v_{n-1})_x}{v_{n-1}} + \frac{\partial}{\partial x}\right)\right] \\ &\cdots\left[\sqrt{\frac{f}{h}}\left(-\frac{(v_1)_x}{v_1} + \frac{\partial}{\partial x}\right)\right](\Psi),\end{aligned} \tag{16}$$

where $L = L(t)$ is an arbitrary function.

3.3. *Reality condition*

Suppose that the coefficients f, g, h in equation (8) and the function v from (11) are all real valued. Then the function U in (11) is real valued, if L satisfies the reality condition

$$\mathrm{i}\frac{L'}{L} = -\frac{\mathrm{Im}(V)}{v'h} + \frac{1}{v'}\sqrt{\frac{f}{h}}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}\frac{W_{n,(u_j)}}{W^*_{n,(u_j)}}\right)\right]_x\right]_x - \frac{\mathrm{i}n}{2v'}\left(-\frac{v''}{v'} + \left[\log\left(\frac{f}{h}\right)\right]_t\right). \tag{17}$$

If this condition is satisfied, then the function U in (11) can be written in the form

$$U = \mathrm{Re}(V) - \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}\left|W_{n,(u_j)}\right|^2\right)\right]_x\right]_x + R,$$

where R is a real-valued function. Note that there need not be a solution to the reality condition (17), as in general its right-hand side depends on x, whereas the left-hand side does not.

4. Reduction to the conventional case

Let us verify that our results from the previous section simplify correctly if the generalized Schrödinger equation (8) is taken to be a known special case. We have the following specifications:

- The conventional time-dependent Schrödinger equation: $f = h = 1$ and $g = 0$.
- Equation (5): $f = 1$, $g = 2\mathrm{i}R$ and $h = 1$.
- Equation (6): $f = 1/(2m)$, $g = -m_x/(2m^2)$ and $h = 1$.
- Equation (7): $f = 1$, $g = 0$ and h arbitrary. Note that here we mean the time-dependent equation that is associated with its stationary case (7).

It is straightforward to verify our results from the previous section in each of the above special cases. In order to keep it short, we do this verification only for the conventional Schrödinger equation, corresponding to the settings $f = h = 1$ and $g = 0$.

Darboux transformation. The Darboux transformation (9) simplifies to

$$\mathcal{D}_{n,(u_j)}(\Psi) = L\frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}, \tag{18}$$

which coincides with the known results [2]. The transformation function F as given in (12) takes the form

$$F = -\int \left(\mathrm{i}\frac{v''}{4v'}x\right)\mathrm{d}x,$$

implying that F_{xx} depends purely on the variable t. Consequently, the last term in the transformed potential (11) depends purely on t and can therefore be absorbed in the function L. In total, the potential U as given in (11) becomes

$$U = V + \mathrm{i}\frac{L'}{L} - 2\left[\log\left(W_{n,(u_j)}\right)\right]_{xx}, \tag{19}$$

where we absorbed the coefficient v' into the arbitrary constant L. It is immediate to see that expression (19) coincides with the well-known transformed potential arising from the conventional Darboux transformation [2].

Chains of Darboux transformation and factorization. For $f = h = 1$ and $g = 0$ the factorization (16) reduces trivially to its well-known counterpart, as each square root becomes equal to one.

Reality condition. Consider the reality condition (17), that we multiply by v' and absorb the latter quantity in L. Next, we apply $f = h = 1$ and $g = 0$, which gives $f/h = 1$ with a vanishing derivative. In total, the reality condition (17) reduces to

$$\mathrm{i}\frac{L'}{L} = -\mathrm{Im}(V) + \left(\frac{W_{n,(u_j)}}{W^*_{n,(u_j)}}\right)_{xx} + \frac{\mathrm{i}nv''}{2(v')^2}.$$

Finally the second term on the right-hand side can be absorbed into L, which yields the known reality condition for the Darboux transformation of the conventional Schrödinger equation [2].

5. Proof of results

We now derive our results stated in section 3.

5.1. Darboux transformation

In the following, we outline how our generalized Darboux transformation will be constructed. Let TDSE and $\mathrm{TDSE_{gen}}$ denote the conventional Schrödinger equation and its generalized counterpart (8), respectively. Assume that there is an invertible point transformation P that takes the generalized Schrödinger equation (8) into its conventional form. We then construct our generalized Darboux transformation $\mathcal{D}_{n,(u_j)}$ by first converting the generalized Schrödinger equation into its conventional form, then applying the conventional Darboux transformation $D_{n,(u_j)}$, and finally reinstalling the generalized form. This procedure results in the following commutative diagram.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{TDSE} \arrow[r, "\substack{\text{Darboux transformation}\\ D_{n,(u_j)}}"] & \mathrm{TDSE} \arrow[d, "\substack{\text{Point}\\ \text{transformation}\\ P^{-1}}"] \\
\mathrm{TDSE}_{\mathrm{gen}} \arrow[u, "\substack{\text{Point}\\ \text{transformation}\\ P}"] \arrow[r, "\mathcal{D}_{n,(u_j)}"] & \mathrm{TDSE}_{\mathrm{gen}}
\end{tikzcd}
$$

In summary, we have the following relation between the conventional and the generalized Darboux transformation:

$$\mathcal{D}_{n,(u_j)} = P^{-1} \circ D_{n,(u_j)} \circ P. \tag{20}$$

We will now construct the point transformation P and use it to calculate (20) explicitly. To this end, consider the generalized Schrödinger equation (8), to which we now apply the following point transformation, introducing a function $F = F(x,t)$ and new coordinates $u = u(x,t)$, $v = v(t)$:

$$\Psi(x,t) = \exp(F(x,t))\Phi(u(x,t), v(t)). \tag{21}$$

This transformation converts the generalized Schrödinger equation to

$$\begin{aligned} \mathrm{i}\Phi_v + \left(\frac{f u_x^2}{v' h}\right)\Phi_{uu} + \frac{1}{v'h}(2F_x f u_x + g u_x + f u_{xx} + \mathrm{i} h u_t)\Phi_u \\ + \frac{1}{v'h}\left(\mathrm{i}F_t h + F_x^2 f + F_{xx} + F_x g - V\right)\Phi = 0. \end{aligned} \tag{22}$$

Here v' denotes the derivative of v; note that v must not depend on x [7] in order to preserve linearity of the equation. Now we convert (22) into a conventional Schrödinger equation by requiring that the coefficient of Φ_{uu} is equal to one, and that the coefficient of Φ_u vanishes:

$$\frac{f u_x^2}{v'h} = 1$$

$$2F_x f u_x + g u_x + f u_{xx} + \mathrm{i} h u_t = 0.$$

These conditions can be solved for the free parameters u and F of our point transformation (21):

$$u = \sqrt{v'} \int \sqrt{\frac{h}{f}}\,\mathrm{d}x \tag{23}$$

$$F = -\int \left(\mathrm{i}\frac{h u_t}{2 f u_x} + \frac{g}{2f} + \frac{u_{xx}}{2u_x}\right)\mathrm{d}x. \tag{24}$$

The new coordinate v remains arbitrary. Now, on plugging the settings (23) and (24) into equation (22), we obtain

$$\mathrm{i}\Phi_v + \Phi_{uu} + \frac{1}{v'h}\big(\mathrm{i}F_t h + F_x^2 f + F_{xx} + F_x g - V\big)\Phi = 0, \tag{25}$$

where the explicit form of F is given in (24). Note that the coefficient of Φ in (25) is still written in the old coordinates x and t. Equation (25) is of the Schrödinger form, such that the Darboux transformation becomes applicable. The Darboux operator D_n of order n, applied to Φ, reads

$$D_{n,(v_j)}(\Phi) = l\,\frac{W_{n,(v_j),\Phi}}{W_{n,(v_j)}}, \tag{26}$$

where $l = l(v)$ is an arbitrary function, (v_j) is a family of n auxiliary solutions of (25), and $W_{n,(v_j),\Phi}$, $W_{n,(v_j)}$ are the nth order Wronskians of the auxiliary solution family (v_j) and of the solution Ψ of equation (25). The function $D_n(\Phi)$ solves the Schrödinger equation

$$\mathrm{i}D_{n,(v_j)}(\Phi)_v + D_{n,(v_j)}(\Phi)_{uu} + \frac{1}{v'h}\big(\mathrm{i}F_t h + F_x^2 f + F_{xx} + F_x g - U\big)D_{n,(v_j)}(\Phi) = 0, \tag{27}$$

where the transformed potential function U reads

$$U = V + \mathrm{i}v'h\frac{l'}{l} - 2v'h\left[\log\big(W_{n,(v_j)}\big)\right]_{uu}. \tag{28}$$

Note that the unusual factor $v'h$ cancels with the same factor in the coefficient of $D_{n,(v_j)}(\Phi)$ in (27). Clearly, here V is understood to be expressed in the new variables u and v. The task is now to rewrite the Darboux transformation (26) and the transformed potential (28) in the variables x and t. Starting with the Darboux transformation, we need to know how the Wronskians transform under the inverse of the point transformation (21). Let (u_j) be a family of n auxiliary solutions of the generalized Schrödinger equation (8) that is related to the family (v_j) via the point transformation (21). We then have [6]

$$\begin{aligned} W_{n,(v_j),\Phi}(u,v) &= \exp(-(n+1)F(x,t))\left(\frac{1}{u_x(x,t)}\right)^{\frac{1}{4}n(n+1)} W_{n,(u_j),\Psi}(x,t),\\ W_{n,(v_j)}(u,v) &= \exp(-nF(x,t))\left(\frac{1}{u_x(x,t)}\right)^{\frac{1}{4}n(n-1)} W_{n,(u_j)}(x,t). \end{aligned} \tag{29}$$

We employ these results in the Darboux transformation (26), which takes the form

$$D_{n,(v_j)}(\Phi) = l\exp(-F)\left(\frac{1}{u_x}\right)^n \frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}. \tag{30}$$

Since the last function is still a solution of (25), we have to multiply it by $\exp(F)$, so as to invert the multiplicative part of the point transformation (21). After doing so and inserting the explicit form (23) of u, we obtain the final result

$$\begin{aligned} \mathcal{D}_{n,(u_j)}(\Psi) &= \exp(F)\,D_{n,(v_j)}(\Phi)\\ &= L\left(\frac{f}{h}\right)^{\frac{n}{2}} \frac{W_{n,(u_j),\Psi}}{W_{n,(u_j)}}, \end{aligned} \tag{31}$$

where $L = l/\sqrt{v'}$. This coincides with (9), as was to be shown. Next, we determine the transformed potential (28), making use of (29):

$$U = V + \mathrm{i}v'h\frac{L'}{L} - 2v'h\left[\log\left(\exp(-nF)\left(\frac{1}{u_x}\right)^{\frac{1}{2}n(n-1)} W_{n,(u_j)}\right)\right]_{uu}$$

$$= V + \mathrm{i}v'h\frac{L'}{L} - 2v'h\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{f}{hv'}\right) + \log\left(W_{n,(u_j)}\right)\right]_{uu}$$
$$= V + \mathrm{i}v'h\frac{L'}{L} - 2f\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{f}{hv'}\right) + \log\left(W_{n,(u_j)}\right)\right]_{xx}$$
$$+ 2h\left(\frac{f}{h}\right)_x\left[-nF + \frac{1}{2}n(n-1)\log\left(\frac{f}{hv'}\right) + \log\left(W_{n,(u_j)}\right)\right]_x. \tag{32}$$

Before we insert the explicit form of F as given in (24), we cast the potential (32) in a slightly different form:

$$U = V + \mathrm{i}v'h\frac{L'}{L} - 2f\left[\log\left(W_{n,(u_j)}\right)\right]_{xx}$$
$$+ 2h\left(\frac{f}{h}\right)_x\left[\log\left(W_{n,(u_j)}\right)\right]_x - \frac{f}{hv'}\left[-2nF + n(n-1)\log\left(\frac{f}{hv'}\right)\right]_{xx}$$
$$+ h\left(\frac{f}{h}\right)_x\left[-2nF + n(n-1)\log\left(\frac{f}{hv'}\right)\right]_x. \tag{33}$$

With this form of the potential it is easy to see that for the conventional Schrödinger equation with $f = h =$ constant and $g = 0$ only the first line (33) of the transformed potential contributes, while the remaining terms vanish. The transformed potential (32) can also be written in the following compact form that we will use in subsequent considerations:

$$U = V + \mathrm{i}v'h\frac{L'}{L} - 2\sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}} W_{n,(u_j)}\right)\right]_x\right]_x$$
$$+ 2nf\left(F_{xx} + \frac{F_x}{2}\left[\log\left(\frac{f}{h}\right)\right]_t\right), \tag{34}$$

where the constant κ is defined as $\kappa = n(n-1)/2$. Clearly, (34) coincides with the sought expression (11). Let us finally insert the function F as given in (24) into the potential (32). After collecting terms we get the following representation of the potential:

$$U = V + \mathrm{i}v'h\frac{L'}{L} - 2f\left[\log\left(W_{n,(u_j)}\right)\right]_{xx} + 2\left(\frac{f}{hv'}\right)_x\left[\log\left(W_{n,(u_j)}\right)\right]_x$$
$$+ nh\left(\frac{ff_x}{2fh} - \frac{f_x^2}{2fh} - \frac{g_x}{h} + \frac{gh_x}{2h^2} - \frac{f_xh_x}{h^2} + \frac{3fh_x^2}{2h^3} + \frac{f_{xx}}{h} - \frac{fh_{xx}}{h^2}\right)$$
$$+ \frac{n^2}{2}\left(\frac{f_x^2}{2f} + \frac{f_xh_x}{h} - \frac{3fh_x^2}{2h^2} - f_{xx} + \frac{fh_{xx}}{h}\right) + \frac{inh}{2}\left(-\frac{v''}{v'} + \frac{f_t}{f} - \frac{h_t}{h}\right). \tag{35}$$

Thus, we have constructed the Darboux transformation for the generalized Schrödinger equation (8).

5.2. *Chains of Darboux transformations and factorization*

It is well known [1] that the factorization property (15) holds for the Schrödinger equation (25), where the form of its potential does not matter here. More explicitly, if we take the Darboux transformation (26) for equation (25), let (v_j) be a family of auxiliary solutions for it and apply it to a further solution Φ of (25), then we have

$$D_{n,(w_j)}(\Phi) = D_{1,s_n} \circ D_{1,s_{n-1}} \circ \cdots \circ D_{1,s_1}(\Phi), \tag{36}$$

where s_j, $j = 1, \ldots, n$, is an auxiliary solution of the $(j-1)$th transformed Schrödinger equation. Now we apply the inverse of the point transformation (21), (23), (24) to relation

(36). We already know the effect of this transformation on the Darboux transformation $D_{n,(v_j)}(\Phi)$ from (30) and (31). Applying the latter two relations to each factor in (36), we get

$$\mathcal{D}_{n,(u_j)}(\Psi) = \mathcal{D}_{1,v_n} \circ \mathcal{D}_{1,v_{n-1}} \circ \cdots \circ \mathcal{D}_{1,v_1}(\Psi), \tag{37}$$

where the families (u_j), (v_j) and the solution Ψ are related to the families (w_j), (s_j) and the solution Φ via the inverse of the point transformation (21), (23), (24), respectively. Clearly, (37) coincides with the sought expression (15).

5.3. Reality condition

We now prove the reality condition (17). To this end, let us consider the transformed potential in its form (34). On employing the explicit form of F from (24), the last term on the right-hand side of (34) reads

$$2nf\left(F_{xx} + \frac{F_x}{2}\left[\log\left(\frac{f}{h}\right)\right]_x\right) = nh\left(\frac{ff_x}{2fh} - \frac{f_x^2}{2fh} - \frac{g_x}{h} + \frac{gh_x}{2h^2} - \frac{f_xh_x}{h^2} + \frac{3fh_x^2}{2h^3} + \frac{f_{xx}}{h} - \frac{fh_{xx}}{h^2}\right) + \frac{\mathrm{i}nh}{2}\left(-\frac{v''}{v'} + \frac{f_t}{f} - \frac{h_t}{h}\right). \tag{38}$$

Now we take our reality condition (17) for L, multiplied by $v'h$:

$$\mathrm{i}v'h\frac{L'}{L} = -\mathrm{Im}(V) + \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}\frac{W_{n,(u_j)}}{W^*_{n,(u_j)}}\right)\right]_x\right]_x - \frac{\mathrm{i}nh}{2}\left(-\frac{v''}{v'} + \left[\log\left(\frac{f}{h}\right)\right]_t\right), \tag{39}$$

and substitute it into the form (34) of the transformed potential. Since

$$\left[\log\left(\frac{f}{h}\right)\right]_t = \frac{f_t}{f} - \frac{h_t}{h},$$

the imaginary part of the last term in (34) will cancel out with the last term on the right-hand side of (39). Let us abbreviate

$$R = \mathrm{Re}\left(2nf\left(F_{xx} + \frac{F_x}{2}\left[\log\left(\frac{f}{h}\right)\right]_t\right)\right),$$

then after insertion of (39) the potential (34) is converted into

$$\begin{aligned} U &= \mathrm{Re}(V) + \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}\frac{W_{n,(u_j)}}{W^*_{n,(u_j)}}\right)\right]_x\right]_x \\ &\quad - 2\sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}W_{n,(u_j)}\right)\right]_x\right]_x + R \\ &= \mathrm{Re}(V) + \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}W_{n,(u_j)}\right)\right]_x\right]_x \\ &\quad - \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}W^*_{n,(u_j)}\right)\right]_x\right]_x \\ &\quad - 2\sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}}W_{n,(u_j)}\right)\right]_x\right]_x + R \end{aligned}$$

$$= \mathrm{Re}(V) - \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}} W_{n,(u_j)}\right)\right]_x\right]_x$$
$$- \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}} W^*_{n,(u_j)}\right)\right]_x\right]_x + R$$
$$= \mathrm{Re}(V) - \sqrt{fh}\left[\sqrt{\frac{f}{h}}\left[\log\left(\left(\frac{f}{hv'}\right)^{\frac{\kappa}{2}} \left|W_{n,(u_j)}\right|^2\right)\right]_x\right]_x + R.$$

This last expression is real valued, and we have proved the validity of the reality condition (17).

6. Application

A particularly interesting special case of the generalized Schrödinger equation (8) is obtained for $f = 1, g = 0$ [10]:

$$\mathrm{i}h\Psi_t + \Psi_{xx} - V\Psi = 0. \tag{40}$$

In the stationary case, after separation of the variable t, we obtain from (40) a Schrödinger equation with a nonconstant factor (weight) in front of the stationary energy. Equivalently, this equation can be seen as a Schrödinger equation with linearly energy-dependent potential. We will now consider a simple, specific case of equation (40), that is,

$$h = \frac{k^2}{qx^2} \tag{41}$$

$$V = \frac{k^2}{x^2} - V_0, \tag{42}$$

where V_0, k are real, positive constants, and $q = q(t)$ is a purely time-dependent, arbitrary function. A particular solution of (40) for the settings (41) and (42) is given by

$$\Psi = \cos(\sqrt{V_0}x)\exp\left(-\mathrm{i}\int q\,\mathrm{d}t\right). \tag{43}$$

The first-order Darboux transformation for (40) with $f = 1, g = 0$, (41) and (42) can be extracted from (13):

$$\mathcal{D}_{1,u_1}(\Psi) = L\sqrt{\frac{1}{h}}\left(-\frac{(u_1)_x}{u_1}\Psi + \Psi_x\right). \tag{44}$$

We choose Ψ to be (43) and the auxiliary solution u_1 we fix as

$$u_1 = \sin(\sqrt{V_0}x)\exp\left(-\mathrm{i}\int q(t)\,\mathrm{d}t\right). \tag{45}$$

We are now ready to perform the first-order Darboux transformation (44) after the insertion of $f = 1, g = 0$ and the functions (43) and (45):

$$\mathcal{D}_{1,u_1}(\Psi) = -L\frac{\sqrt{qV_0}x}{k\sin(\sqrt{V_0}x)}\exp\left(-\mathrm{i}\int q\,\mathrm{d}t\right),$$

where several terms depending on the variable t have been absorbed into L. The function $\mathcal{D}_{1,u_1}(\Psi)$ solves equation (40) with h as given in (41) and the potential V replaced by its transformed counterpart, that we obtain by substituting (41) and (42) into (11):

$$U = \mathrm{i}\frac{k^2L'}{qLx^2} - V_0 + \frac{k^2}{x^2} - \frac{2\sqrt{V_0}}{x}\cot(\sqrt{V_0}x) + \frac{2V_0}{\sin^2(\sqrt{V_0}x)} + \frac{q'}{2qx} + \mathrm{i}\frac{(q')^2k^2\log(x)}{4q^3x}. \tag{46}$$

Note that for the sake of simplicity we took $v(t) = t$. Let us finally evaluate the reality condition (17), we have in the present case with $n = 1$:

$$L = \exp\left(-\frac{1}{4}\int\left(\frac{q'}{q}\right)^2 x\log(x)\,\mathrm{d}x\right).$$

In order to substitute this expression into the potential (46) we calculate

$$\frac{L'}{L} = -\frac{(q')^2 x\log(x)}{4q^2}.$$

If we insert this into the first term on the right-hand side of (46), it is immediately clear that all imaginary terms cancel out, such that the transformed potential becomes real valued.

7. Concluding remarks

We have shown that a generalized time-dependent Schrödinger equation with first derivative terms and arbitrary, independent coefficients always admits a Darboux transformation. This generalized Darboux transformation maintains fundamental properties of the conventional Darboux transformation, such as factorization and a reality condition. Thus, the present paper explains the existence and gives the explicit form of the particular Darboux transformations that have been found for several special cases of our generalized Schrödinger equation. Further issues related to our work, such as the supersymmetry formalism and the construction of intertwiners, are the subject of ongoing research.

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